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NOTE ON REPRESENTATIONS OF THE PARTIAL SUM OF A FOURIER'S SERIES.

BY DUNHAM JACKSON.

The problem of the convergence of expansions in series associated with ordinary linear differential equations has been developed with a high degree of generality by Birkhoff* and others. It is a characteristic of the method employed that the sum of the first n terms of the series is represented by a contour integral in the plane of the complex parameter. For the fundamental particular case of Fourier's series, it is a somewhat interesting exercise to derive the ordinary real formula for the partial sum directly from the contour integral, without going back to the individual terms which make up the sum. Of course no new conclusions are reached, and no novel methods are required; the most that is new is the opening of a fresh path through a much-traversed region. Two parallel discussions are given below, corresponding to the alternative systems, one of the first order and one of the second, from which the Fourier development is obtained.

I. System of the first order.

Let us begin with the system composed of the differential equation

$$\frac{du}{dx} + \lambda u = 0$$

and the boundary condition

$$u(0) = u(2\pi).$$

Its characteristic numbers are the numbers $\lambda = ki$, $k = 0, \pm 1, \pm 2, \cdots$, and the corresponding characteristic functions have the form

$$u_k(x) = e^{-kix}.$$

They lead to the expansion of an arbitrary function $f(x)$ in a series of the form

$$\sum_{k=-\infty}^{+\infty} a_k u_k(x),$$

where†

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* Transactions of the American Mathematical Society, vol. 9 (1908), pp. 373–395.

† The value of the integral in the denominator is of course found to be simply 2π, when the explicit expressions for $u_k(t)$ and $v_k(t)$ are inserted.

$$a_k = \frac{\int_0^{2\pi} f(t) v_k(t)\,dt}{\int_0^{2\pi} u_k(t) v_k(t)\,dt},$$

the functions

$$v_k(x) = e^{kix}$$

being the characteristic solutions of the adjoint system

$$-\frac{dv}{dx} + \lambda v = 0, \qquad v(0) = v(2\pi).$$

In this form, the series does not appear strikingly familiar; but on combining the terms whose indices are the negatives of each other, and expressing the exponentials in terms of trigonometric functions, it is found that

$$a_k u_k(x) + a_{-k} u_{-k}(x) = \frac{1}{\pi} \cos kx \int_0^{2\pi} f(t) \cos kt\,dt + \frac{1}{\pi} \sin kx \int_0^{2\pi} f(t) \sin kt\,dt,$$

so that the given system does really lead to the ordinary Fourier's series.*

The Green's function of the system, $G(x, s, \lambda)$, is

$$(1) \qquad \frac{e^{\lambda(s-x)}}{1 - e^{-2\lambda\pi}}, \quad x > s; \qquad \frac{e^{\lambda(s-x-2\pi)}}{1 - e^{-2\lambda\pi}}, \quad x < s.$$

This may be found, for example, by direct substitution in the general determinant formula.† In accordance with the general theory, the sum of the first terms of the series is given by

$$\sum_{k=-n}^{+n} a_k u_k(x) = \frac{1}{2\pi i} \int_0^{2\pi} f(s) \int_C G(x, s, \lambda)\,d\lambda\,ds;$$

the integral is extended around a contour in the λ-plane presently to be specified. Our problem, then, may be stated in the following form, which is intelligible on its own merits, quite apart from the general considerations that have led up to it:

To establish the identity

$$(2) \qquad \frac{1}{2\pi i} \int_C G(x, s, \lambda)\,d\lambda = \frac{1}{\pi} \frac{\sin (n + \frac{1}{2})(s - x)}{2 \sin \frac{1}{2}(s - x)},$$

where G is the function defined by (1), *and C is a contour inclosing the points* $0, \pm i, \pm 2i, \cdots, \pm ni$, *in the λ-plane, but no other roots of the denominator of G.*

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* Cf. Birkhoff, loc. cit., p. 389.

† Cf. Birkhoff, loc. cit., p. 378.

Beyond the requirement just stated, the exact form of the contour is without effect on the value of the integral, since the integrand is analytic except where the denominator vanishes; but it does have an influence on the ease of actually performing the calculation. We shall find it convenient to use for C the rectangle having its vertices at the points $\pm g \pm (n + \frac{1}{2})i$, where n has the value which appears in (2), and g is an arbitrary positive real quantity, which will be allowed to increase without limit in the course of the proof. We shall let M, N, P, Q denote the corners of the rectangle in the first, second, third and fourth quadrants respectively.

For our purposes, no significance attaches to the variables s and x separately; we may introduce a single letter for the quantity

$$s - x = t.$$

Then we have to consider separately the cases that $-2\pi < t < 0$, $0 < t < 2\pi$. It appears that no provision is made hereby for the values $t = 0, \pm 2\pi$; but with regard to these, it suffices to remark that the integral which forms the left-hand member of (2) is obviously a continuous function of t.

Suppose first that $t < 0$.

On the upper side MN of the rectangle, we may set*

$$\lambda = u + (n + \tfrac{1}{2})i;$$

the imaginary part is constant, and the real part u varies from $+g$ to $-g$. On making the substitution in the integral, it appears that

$$\int_M^N \frac{e^{\lambda t}}{1 - e^{-2\lambda\pi}}\, d\lambda = e^{(n+\frac{1}{2})it} \int_{+g}^{-g} \frac{e^{ut}}{1 + e^{-2u\pi}}\, du.$$

In the same way,

$$\int_P^Q \frac{e^{\lambda t}}{1 - e^{-2\lambda\pi}}\, d\lambda = e^{-(n+\frac{1}{2})it} \int_{-g}^{+g} \frac{e^{ut}}{1 + e^{-2u\pi}}\, du.$$

So the integral about the whole contour has the expression

$$(3) \quad \int_C G d\lambda = [e^{-(n+\frac{1}{2})it} - e^{(n+\frac{1}{2})it}] \int_{-g}^{+g} \frac{e^{ut}}{1 + e^{-2u\pi}}\, du + \int_Q^M G d\lambda + \int_N^P G d\lambda.$$

The right-hand member may be replaced by its limit for $g = \infty$, since its value is independent of g. We shall see that the second and third integrals separately approach zero; it will follow that the first term approaches a limit as g becomes infinite (a fact which is readily established directly), and that the value of this limit is the value of the contour integral.

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* The use of the letter u here is independent of that in the first few lines of the section.

Consider the side QM. If we set

$$\lambda = g + vi, \quad -(n + \tfrac{1}{2}) \leqq v \leqq n + \tfrac{1}{2},$$

then

$$| e^{-2\lambda\pi} | = e^{-2g\pi},$$

and

$$| 1 - e^{-2\lambda\pi} | \geqq 1 - e^{-2g\pi} > \tfrac{1}{2},$$

if g is sufficiently large. On the other hand,

$$| e^{\lambda t} | = e^{gt} = e^{-g|t|},$$

and this approaches zero as g becomes infinite, so that the first member approaches zero uniformly throughout the finite range of variation of v, for any particular value of t. The same thing therefore is true of G, and the statement with regard to the second integral on the right of (3) is justified. By writing

$$G = \frac{e^{\lambda t}}{1 - e^{-2\lambda\pi}} = \frac{e^{\lambda(t+2\pi)}}{e^{2\lambda\pi} - 1},$$

and setting

$$\lambda = -g + vi$$

along the side NP, we may apply similar reasoning to the third term. We recognize thus that

$$\int_C G d\lambda = -2i \sin \{(n + \tfrac{1}{2})t\} \int_{-\infty}^{+\infty} \frac{e^{ut}}{1 + e^{-2u\pi}} du, \tag{4}$$

in which the bracket factor from (3) has been replaced by an equivalent simpler expression. It is noticeable that the index n no longer appears under the sign of integration.

Let

$$J = \int_{-\infty}^{+\infty} \frac{e^{ut}}{1 + e^{-2u\pi}} du, \qquad J_g = \int_{-g}^{+g} \frac{e^{ut}}{1 + e^{-2u\pi}} du.$$

Introducing a new complex variable z, let us consider the integral of the function

$$\frac{e^{zt}}{1 + e^{-2z\pi}}$$

extended in the positive direction around the rectangle in the z-plane whose vertices R, S, T, U are the points $g + i$, $-g + i$, $-g$, $+g$ respectively. The integrand is analytic throughout the rectangle, except at the one point $z = \frac{1}{2}i$. There it has the residue $e^{\frac{1}{2}it}/2\pi$, as is seen by expanding the denominator in powers of $z - \frac{1}{2}i$, and performing the indicated division. The value of the contour integral accordingly is $ie^{\frac{1}{2}it}$.

Taking the sides of the rectangle one by one, we see first that the integral over TU is precisely J_g. On RS, we may set $z = \xi + i$, where ξ is real, and find that

$$\int_R^S \frac{e^{zt}}{1 + e^{-2z\pi}}\, dz = \int_{+g}^{-g} \frac{e^{\xi t} e^{it}}{1 + e^{-2\xi\pi}}\, d\xi = - e^{it} J_g.$$

Then

$$ie^{\frac{1}{2}it} = (1 - e^{it}) J_g + J_g' + J_g'',$$

if the last two terms are understood to represent the integrals over the vertical sides of the rectangle. As at an earlier stage in the demonstration, we proceed to let g become infinite; it appears that J_g' and J_g'' approach zero, and*

$$(5) \qquad ie^{\frac{1}{2}it} = (1 - e^{it}) J, \qquad J = \frac{i}{e^{-\frac{1}{2}it} - e^{\frac{1}{2}it}} = - \frac{1}{2 \sin \frac{1}{2}t}.$$

Combination of (4) and (5) gives for the original contour integral the expression

$$(6) \qquad \int_{C_1} G d\lambda = \frac{i \sin \{(n + \frac{1}{2})t\}}{\sin \frac{1}{2}t};$$

and on division by $2\pi i$ and substitution of the value of t, the desired identity is established, for the case that $- 2\pi < t < 0$.

The other case, that $0 < t < 2\pi$, can be reduced immediately to the preceding. For if we set

$$t' = t - 2\pi = s - x - 2\pi,$$

the variable t' enters into the expression for the Green's function just as t entered before, and its range of variation is the same as that which was assigned to t before. The previous work is valid throughout, with t replaced by t'; and in the final formula (6), since the right-hand member, regarded as a function of t, has the period 2π, the accent may be dropped.

II. System of the second order.

Let us turn to the differential equation

$$\frac{d^2u}{dx^2} + \lambda u = 0,$$

with the pair of boundary conditions

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi).$$

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* Cf. Goursat, Cours d'analyse mathématique, vol. 2, exercise 15 at the end of Chapter XIV.

Instead of λ, we shall use for the most part the parameter ρ, defined by the equation

$$\rho^2 = \lambda.$$

The characteristic solutions are*

$$\cos kx, \qquad \sin kx, \qquad k = 0, 1, 2, \cdots.$$

The given system is self-adjoint, and the expansions associated with it are obtained immediately in the familiar form of Fourier's series. The expressions for the Green's function, when $x > s$ and when $x < s$, are respectively

$$\frac{\cos \rho(\pi + s - x)}{2\rho \sin \rho\pi}, \qquad \frac{\cos \rho(\pi + x - s)}{2\rho \sin \rho\pi}.$$

We shall be concerned with the integral

$$\int_C G d\lambda = \int_{C_1} 2\rho G d\rho,$$

the path of integration being a circuit in the λ-plane, or a half-circuit in the ρ-plane; the exact form which it will be convenient to use will be specified later.

As before, the cases $x > s$ and $x < s$ require discussions that are separate at the start, but speedily coalesce into one. If $x > s$, let $t = \pi + s - x$; if $x < s$, let $t = \pi + x - s$; then t is subject to the inequalities

$$-\pi < t < \pi$$

in either case, and in each case we have to evaluate the integral

$$\int_{C_1} \frac{\cos \rho t}{\sin \rho\pi} d\rho.$$

With regard to the path of integration, let M, N, P, Q be used, independently of the preceding section, to denote respectively the points $-gi$, $n + \frac{1}{2} - gi$, $n + \frac{1}{2} + gi$, gi. Then the integral is to be extended along the sides MN, NP, PQ, but *not* along the fourth side of the rectangle, since, as has been noticed already, a half-circuit in the ρ-plane corresponds to a complete circuit in the λ-plane. The value of the integral is independent of the value of g.

On the vertical side, let

$$\rho = n + \tfrac{1}{2} + vi,$$

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* Of course the value $k = 0$ is to be excluded in the case of the sine.

where v is a real variable which goes from $-g$ to g. The denominator of the integrand assumes the form

$$\sin\{(n + \tfrac{1}{2} + vi)\pi\} = (-1)^n \cos v\pi i.$$

The numerator is

$$\cos\{(n + \tfrac{1}{2} + vi)t\} = \cos\{(n + \tfrac{1}{2})t\} \cos vti - \sin\{(n + \tfrac{1}{2})t\} \sin vti.$$

The factors $\cos\{(n + \frac{1}{2})t\}$, $\sin\{(n + \frac{1}{2})t\}$ are constant as far as the integration is concerned, and may be taken out before the integral sign. In the second term, the remaining factor $\sin vti$ is an odd function of v; and since the denominator $\cos v\pi i$ is even, the whole integrand is odd, and the corresponding term of the integral will vanish. For the significant part of the contour integral, we have the expression

$$\int_N^P \frac{\cos \rho t}{\sin \rho\pi} d\rho = i(-1)^n \cos\{(n + \tfrac{1}{2})t\} \int_{-g}^{+g} \frac{\cos vti}{\cos v\pi i} dv.$$

By setting $\rho = u \pm gi$, for the integration along the horizontal sides MN and PQ, and writing the trigonometric functions in terms of exponentials, it is readily seen that these integrals approach zero as g becomes infinite. It follows that

$$\int_{C_1} \frac{\cos \rho t}{\sin \rho\pi} d\rho = i(-1)^n \cos\{(n + \tfrac{1}{2})t\} \int_{-\infty}^{+\infty} \frac{\cos vti}{\cos v\pi i} dv.$$

The integral which remains in the right-hand member of the last relation can be reduced at once to that which was evaluated at the corresponding stage in the preceding section. In the first place,

$$\cos vti = \tfrac{1}{2}(e^{vt} + e^{-vt}),$$

and

$$\int_{-\infty}^{+\infty} \frac{e^{vt}}{\cos v\pi i} dv = \int_{-\infty}^{+\infty} \frac{e^{-vt}}{\cos v\pi i} dv,$$

as is seen by changing v into $-v$, so that

$$\int_{-\infty}^{+\infty} \frac{\cos vti}{\cos v\pi i} dv = \int_{-\infty}^{+\infty} \frac{e^{vt}}{\cos v\pi i} dv.$$

Then, if $\tau = t - \pi$, the new variable τ has the same range of values as t in the preceding section, and by making use of the earlier work we find that

$$\int_{-\infty}^{+\infty} \frac{e^{vt}}{\cos v\pi i} dv = 2\int_{-\infty}^{+\infty} \frac{e^{v(\tau+\pi)}}{e^{v\pi} + e^{-v\pi}} dv = 2\int_{-\infty}^{+\infty} \frac{e^{vt}}{1 + e^{-2v\pi}} dv = -\frac{1}{\sin \frac{1}{2}\tau}.$$

It remains only to combine and interpret the results that have been obtained. If $t = \pi + s - x$, then

$$\cos (n + \tfrac{1}{2})t = (-1)^{n+1} \sin (n + \tfrac{1}{2})(s - x),$$

$$\sin \tfrac{1}{2}\tau = \sin \tfrac{1}{2}(t - \pi) = \sin \tfrac{1}{2}(s - x),$$

$$\int_C G d\lambda = \int_{C_1} \frac{\cos \rho t}{\sin \rho\pi} d\rho = \frac{i \sin (n + \frac{1}{2})(s - x)}{\sin \frac{1}{2}(s - x)}.$$

If $t = \pi + x - s$, the only difference is that $s - x$ in the last formula is replaced by $x - s$, and this change has no effect on the value of the fraction.

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BY FRANK IRWIN.

Professor Bôcher has called my attention to the fact that the contents of my note, "Relation between the roots of a rational integral function and its derivative," in the March, 1915, number of the *Annals* are substantially identical with the proof given by him in the *Annals*, vol. 7, 1892, page 70.